CHERN CLASSES AND LIE-RINEHART ALGEBRAS

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Abstract. Let A be a F-algebra where F is a field, and let W be an A-module of finite presentation. We use the linear Lie-Rinehart algebra $\mathbf{V}_W$ of W to define the first Chern-class $c_1(W)$ in $\mathrm{H}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$, where U in $\mathrm{Spec}(A)$ is the open subset where W is locally free. We compute explicitly algebraic $\mathbf{V}_W$-connections on maximal Cohen-Macaulay modules W on the hypersurface-singularities $B_{mn2} = x^m + y^n + z^2$, and show that these connections are integrable, hence the first Chern-class $c_1(W)$ vanishes. We also look at indecomposable maximal Cohen-Macaulay modules on quotient-singularities in dimension 2, and prove that their first Chern-class vanish.

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Introduction

Classically, the Chern-classes of a locally free coherent A-module W are defined using the curvature R_∇ of a connection $\nabla : W \to W \otimes \Omega^1_X$. A connection ∇ gives rise to a covariant derivation $\overline{\nabla} : \mathrm{Der}_F(A) \to \mathrm{End}_F(W)$. If we more generally consider a coherent A-module W, a connection ∇ might not exist. In this paper we will consider the problem of defining Chern-classes for A-modules W using a covariant derivation defined on a certain sub-Lie-algebra $\mathbf{V}_W$ of $\mathrm{Der}_F(A)$: For an arbitrary A-module W there exists a sub-Lie-algebra $\mathbf{V}_W$ of $\mathrm{Der}_F(A)$ called the linear Lie-Rinehart algebra, and also the notion of a $\mathbf{V}_W$-connection. There exists a complex, the Chevalley-Hochshild complex $\mathrm{C}^\bullet(\mathbf{V}_W, W)$ for the A-module W with a flat $\mathbf{V}_W$-connection, generalizing the classical deRham-complex. If A is a regular ring, the derivation module $\mathrm{Der}_F(A)$ is locally free, and it follows that the complex $\mathrm{C}^\bullet(\mathrm{Der}_F(A), A)$ is quasi-isomorphic to the complex $\Omega^\bullet_{A/F}$, hence the Chevalley-Hochschild complex of $\mathrm{Der}_F(A)$ can be used to compute the algebraic

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deRham-cohomology of A. The complex $\mathrm{C}^\bullet(\mathbf{V}_W, A)$ generalizes simultaneously the algebraic deRham-complex and the Chevalley-Eilenberg complex. This is due to [22]. A natural thing to do is to investigate possibilities of defining Chern-classes of A-modules equipped with a $\mathbf{V}_W$-connection, generalizing the classical Chern-classes defined using the curvature of a connection. Invariants for Lie-Rinehart algebras have been considered by several authors (see [8], [10] and [14]), however invariants for A-modules with a $\mathbf{V}_W$-connection, where $\mathbf{V}_W$ is the linear Lie-Rinehart algebra of W does not appear to be treated in the litterature and that is the aim of this work. In this paper we also develop techniques to do explicit calculations of Chern-classss of maximal Cohen-Macaulay modules on hypersurface singularities and two-dimensional quotient singularities.

We define for any F-algebra A where F is any field, and any A-module W which is locally free on an open subset U of $\mathrm{Spec}(A)$, the first Chern-class $c_1(W)$ in $\mathrm{H}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$, where $\mathrm{H}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$ is the Chevalley-Hochschild cohomology of the restricted linear Lie-Rinehart algebra $\mathbf{V}_W$ of W, with values in the sheaf $\mathcal{O}_U$. This is Theorem 3.2. We also prove in Theorem 2.1 existence of explicit $\mathbf{V}_W$-connections $\overline{\nabla}^{\psi,\phi}$ on a class of maximal Cohen-Macaulay modules W on the Brieskorn singularities B_{mn2}, which in fact are defined over any field F of characteristic prime to m and n. We prove in Theorem 3.3 that the $\mathbf{V}_W$-connections defined in Theorem 2.1 are all regular, hence the first Chern-class is zero. Finally we prove in Theorem 4.2 that for any maximal Cohen-Macaulay module W_ρ on any two-dimensional quotient-singularity $\mathbf{C}^2/G$, the first chern class $c_1(W_\rho)$ is zero.

1. Kodaira-Spencer maps and linear Lie-Rinehart algebras

Let A be an F-algebra, where F is any field, and let W be an A-module. in this section we use the Kodaira-Spencer class and the Kodaira-Spencer map to define the linear Lie-Rinehart algebra $\mathbf{V}_W$ of W, and the obstruction $lc(W)$ in $\mathrm{Ext}^1_A(\mathbf{V}_W, \mathrm{End}_A(W))$ for existence of a $\mathbf{V}_W$-connection on W.

Definition 1.1. Let P be an A-bimodule. The *Hochschild-complex* of A with values in P $\mathrm{C}^\bullet(A, P)$ is defined as follows:

$$\mathrm{C}^p(A, P) = \mathrm{Hom}_F(A^{\otimes p}, P)$$

with differentials $d^p : \mathrm{C}^p(A, P) \to \mathrm{C}^{p+1}(A, P)$ defined by

$$d^p\phi(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = a_1\phi(a_2 \otimes \cdots \otimes a_{p+1})+$$

$$\sum_{1\leq i\leq p} (-1)^i\phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) + (-1)^{p+1}\phi(a_1 \otimes \cdots \otimes a_p)a_{p+1}.$$

We adopt the convention that $\mathrm{C}^0(A, P) = P$, and $d^0(p)(a) = pa - ap$ for all p in P, and a in A. The i'th cohomology $\mathrm{H}^i(\mathrm{C}^\bullet(A, P))$ is denoted $\mathrm{HH}^i(A, P)$.

There exists an exact sequence

$$0 \to \mathrm{HH}^0(A, P) \to P \to \mathrm{Der}_F(A, P) \to \mathrm{HH}^1(A, P) \to 0, \tag{1.1.1}$$

and it is well known that if we let $P = \mathrm{Hom}_F(W, V)$ where W and V are A-modules, then $\mathrm{HH}^i(A, P)$ equals $\mathrm{Ext}^i_A(W, V)$. Recall the definition of a *connection* on W: it is an F-linear map $\nabla : W \to W \otimes \Omega^1_{A/F}$ with the property that $\nabla(aw) = a\nabla(w) + w \otimes da$, where $d : A \to \Omega^1_{A/F}$ is the *universal derivation.* Put $P = \mathrm{Hom}_F(W, W \otimes \Omega^1_{A/F})$

in 1.1.1, and construct an element C in $\mathrm{Der}_F(A, \mathrm{Hom}_F(W, W \otimes \Omega^1_{A/F}))$ in the following way: $C(a)(w) = w \otimes da$.

Definition 1.2. The class $\overline{C} = ks(W)$ in $\mathrm{Ext}^1_A(W, W \otimes \Omega^1_{A/F})$ is the *Kodaira-Spencer class* of W.

Note that $ks(W)$ is also referred to as the *Atiyah-class* of W.

Proposition 1.3. *Let A be any F-algebra, and let W be any A-module, then $ks(W) = 0$ if and only if W has a connection.*

Proof. We see that $ks(W) = 0$ if and only if there exists an element

$$\nabla : W \to W \otimes \Omega^1_{A/F}$$

with the property that $d^0\nabla = C$. This is if and only if

$$(\nabla a - a\nabla)(w) = \nabla(aw) - a\nabla(w) = C(a)(w) = w \otimes da,$$

hence ∇ is a connection, and the claim follows. □

Definition 1.4. Let A be an F-algebra where F is any field. A *Lie-Rinehart algebra* on A is a F-Lie-algebra and an A-module $\mathfrak{g}$ with a map $\alpha : \mathfrak{g} \to \mathrm{Der}_F(A)$ satisfying the following properties:

$$\alpha(a\delta) = a\alpha(\delta) \tag{1.4.1}$$

$$\alpha([\delta, \eta]) = [\alpha(\delta), \alpha(\eta)] \tag{1.4.2}$$

$$[\delta, a\eta] = a[\delta, \eta] + \alpha(\delta)(a)\eta \tag{1.4.3}$$

for all $a \in A$ and $\delta, \eta \in \mathfrak{g}$. Let W be an A-module. A $\mathfrak{g}$-*connection* ∇ on W, is an A-linear map $\nabla : \mathfrak{g} \to \mathrm{End}_F(W)$ wich satisfies the *Leibniz-property*, i.e.

$$\nabla(\delta)(aw) = a\nabla(\delta)(w) + \alpha(\delta)(a)w$$

for all $a \in A$ and $w \in W$. We say that (W, ∇) is a $\mathfrak{g}$-*module* if ∇ is a homomorphism of Lie-algebras. The *curvature* of the $\mathfrak{g}$-connection, R_∇ is defined as follows:

$$R_\nabla(\delta \wedge \eta) = [\nabla_\delta, \nabla_\eta] - \nabla_{[\delta,\eta]}.$$

Example 1.5. *Any connection ∇ on W, gives an action $\nabla : \mathrm{Der}_F(A) \to \mathrm{End}_F(W)$ with the property that $\nabla(\delta)(aw) = a\nabla(\delta)(w) + \delta(a)w$ for any δ in $\mathrm{Der}_F(A)$, a in A and w in W. The Lie-algebra $\mathfrak{g} = \mathrm{Der}_F(A)$ is trivially a Lie-Rinehart algebra, hence W has a $\mathfrak{g}$-connection.*

Given any Lie-algebroid $\mathfrak{g}$, and any A-module W with a $\mathfrak{g}$-connection, the set of $\mathfrak{g}$-connections on W is a *torsor* on the set $\mathrm{Hom}_A(\mathfrak{g}, \mathrm{End}_A(W))$. Put $P = \mathrm{Hom}_F(W, W)$ in 1.1.1, and define for all δ in $\mathrm{Der}_F(A)$ the following element $C(\delta)$ in $\mathrm{Der}_F(A, \mathrm{Hom}_F(W, W))$: $C(\delta)(a)(m) = \delta(a)m$. We get an A-linear map

$$C : \mathrm{Der}_F(A) \to \mathrm{Der}_F(A, \mathrm{Hom}_F(W, W)).$$

Definition 1.6. Let A be any F-algebra, and let W be any A-module. We define the *Kodaira-Spencer map*

$$g : \mathrm{Der}_F(A) \to \mathrm{Ext}^1_A(W, W)$$

as follows: $g(\delta) = \overline{C(\delta)}$ in sequence 1.1.1. We let $ker(g) = \mathbf{V}_W$ be the *linear Lie-Rinehart algebra* of W.

One immediately checks that the A-sub module $\mathbf{V}_W$ of $\mathrm{Der}_F(A)$ satisfies the axioms of definition 1.4, hence $\mathbf{V}_W$ is indeed a Lie-Rinehart algebra.

Proposition 1.7. *Let A be any F-algebra and W any A-module. There exists an F-linear map*

$$\rho : \mathbf{V}_W \to \mathrm{End}_F(W)$$

with the property that $\rho(\delta)(aw) = a\rho(\delta)(w) + \delta(a)w$ for all δ in $\mathbf{V}_W$, a in A and w in W.

Proof. Assume that $g(\delta) = 0$. Then there exists a map $\rho(\delta)$ in $\mathrm{Hom}_F(W, W)$ with the property that $d^0\rho(\delta) = C(\delta)$. This is if and only if $\rho(\delta)(aw) = a\rho(\delta)(w)+\delta(a)w$, hence for all δ in $\mathbf{V}_W$ we get a map $\rho(\delta)$, and the assertion follows. □

Given any A-module W, we now pick any map $\rho : \mathbf{V}_W \to \mathrm{End}_F(W)$ with the property that $\rho(\delta)(aw) = a\rho(\delta)(w) + \delta(a)w$, which exists by proposition 1.7. Put $P = \mathrm{Hom}_F(\mathbf{V}_W, \mathrm{End}_A(W))$ in sequence 1.1.1 and consider the element L in $\mathrm{Der}_F(A, P)$ defined as follows: $L(a)(\delta)(w) = a\rho(\delta)(w) - \rho(a\delta)(w)$.

Definition 1.8. Let $lc(W) = \overline{L}$ in $\mathrm{HH}^1(A, P) = \mathrm{Ext}^1_A(\mathbf{V}_W, \mathrm{End}_A(W))$.

One verifies that the class $lc(W)$ is independent of choice of map ρ, hence it is an invariant of W.

Theorem 1.9. *Let A be any F-algebra, and W any A-module, then $lc(W) = 0$ if and only if W has a $\mathbf{V}_W$-connection.*

Proof. Assume $lc(W) = 0$. Then there exists a map η in $\mathrm{Hom}_F(\mathbf{V}_W, \mathrm{End}_A(W))$ with the property that $d^0\eta = L$. Then the map $\rho + \eta = \nabla : \mathbf{V}_W \to \mathrm{End}_A(W)$ is a $\mathbf{V}_W$-connection, and the assertion follows. □

From a groupoid in schemes (roughly speaking an algebraic stack, see [16]) one constructs a Lie-Rinehart algebra in a way similar to the way one constructs the Lie-algebra from a group-scheme. A natural problem is to find necessary and sufficient criteria for the linear Lie-Rinehart algebra to be integrable to a groupoid in schemes.

Note that for any A-submodule and k-sub-Lie algebra $\mathfrak{g}$ of $\mathrm{Der}_k(A)$, there exists a *generalized universal enveloping algebra* $\mathrm{U}(A, \mathfrak{g})$ which is a sub-algebra of $\mathrm{D}(A)$, where $\mathrm{D}(A)$ is the ring of differential-operators of A. The algebra $\mathrm{U}(A, \mathfrak{g})$ has the property that there is a one-to-one correspondence between A-modules with a flat $\mathfrak{g}$-connection and left $\mathrm{U}(A, \mathfrak{g})$-modules. There exists a generalized PBW-theorem for the algebra $\mathrm{U}(A, \mathfrak{g})$ when $\mathfrak{g}$ is a projective A-module (see [23]). The dual algebra $\mathrm{U}(A, \mathfrak{g})^*$ is commutative and its spectrum $\mathrm{Spec}(\mathrm{U}(A, \mathfrak{g})^*)$ is a formal equivalence-relation in schemes. Note also that all constructions in this section globalize.

2. Explicit examples: Algebraic $\mathbf{V}_W$-connections

In this section we apply the theory developed in the previous section to compute explicitly algebraic $\mathbf{V}_W$-connections on a class of maximal Cohen-Macaulay modules on isolated hypersurface singularities $B_{mn2} = x^m + y^n + z^2$. Let in the following F be a field of characteristic zero, and $A = F[[x, y, z]]/x^m + y^n + z^2$. We are interested in maximal Cohen-Macaulay modules on A, and such modules have

a nice description, due to [6]: Consider the two matrices

$$\phi = \begin{pmatrix} x^{m-k} & y^{n-l} & 0 & z \\ y^l & -x^k & z & 0 \\ z & 0 & -y^{n-l} & -x^k \\ 0 & z & x^{m-k} & -y^l \end{pmatrix}$$

and

$$\phi = \begin{pmatrix} x^k & y^{n-l} & z & 0 \\ y^l & -x^{m-k} & 0 & z \\ 0 & z & -y^l & x^k \\ z & 0 & -x^{m-k} & -y^{n-l} \end{pmatrix},$$

where $1 \leq k \leq m$ and $1 \leq l \leq n$. Let f be the polynomial $x^m + y^n + z^2$. The matrices ϕ and ψ have the property that $\phi\psi = \psi\phi = fI$ where I is the rank 4 identity matrix. Hence we get a complex of A-modules

$$\cdots \to^{\psi} P \to^{\phi} P \to^{\psi} P \to^{\phi} P \to W(\phi, \psi) \to 0. \tag{2.0.1}$$

Note that the sequence 2.0.1 is a complex since $\phi\psi = \psi\phi = fI = 0$. By [6], the module $W = W(\phi, \psi)$ is a maximal Cohen-Macaulay module on A. The ordered pair (ϕ, ψ) is a *matrix-factorization* of the polynomial f. We want to compute explicitly algebraic $\mathbf{V}_W$-connections on the modules $W = W(\phi, \psi)$ for all $1 \leq k \leq m$ and $1 \leq l \leq n$, i.e we want to give explicit formulas for A-linear maps $\nabla^{\phi,\psi} = \nabla : \mathbf{V}_W \to \mathrm{End}_F(W)$ satisfying $\nabla(\delta)(aw) = a\nabla(\delta)(w) + \delta(a)w$ for all a in A, w in W and δ in $\mathbf{V}_W$. Hence first we have to compute generators and syzygies of the derivation-modules $\mathrm{Der}_F(A)$. A straight-forward calculation shows that $\mathrm{Der}_F(A)$ is generated by the derivations

$$\begin{aligned} \delta_0 &= 2nx\partial_x + 2my\partial_y + mnz\partial_z \\ \delta_1 &= mx^{m-1}\partial_y - ny^{n-1}\partial_z \\ \delta_2 &= -2z\partial_x + mx^{m-1}\partial_z \\ \delta_3 &= -2z\partial_y + ny^{n-1}\partial_z, \end{aligned}$$

hence we get a surjective map of A-modules $\eta : A^4 \to \mathrm{Der}_F(A) \to 0$. A calculation showas that the syzygy-matrix of $\mathrm{Der}_F(A)$ is the following matrix

$$\rho = \begin{pmatrix} y^{n-1} & z & 0 & x^{m-1} \\ 2x & 0 & -2z & -2y \\ 0 & nx & ny^{n-1} & -nz \\ -mz & my & -mx^{m-1} & 0 \end{pmatrix},$$

hence we get an exact sequence of A-modules

$$\cdots \to A^4 \to^{\rho} A^4 \to^{\eta} \mathrm{Der}_F(A) \to 0.$$

A calculation shows that the Kodaira-Spencer map $g : \mathrm{Der}_F(A) \to \mathrm{Ext}^1_A(W, W)$ is zero for the modules W, hence $\mathbf{V}_W = \mathrm{Der}_F(A)$, and the calculation also provides us with elements $\nabla(\delta_i)$ in $\mathrm{End}_F(W)$ with the property that $\nabla(\delta_i)(aw) = a\nabla(\delta_i)(w) + \delta_i(a)w$ for $i = 0, .., 3$. Hence we get an F-linear map $\nabla : \mathbf{V}_W \to \mathrm{End}_F(W)$. Given any map e_i in $\mathrm{End}_A(W)$, it follows that the map $\nabla(\delta_i) + e_i$ again is an element in $\mathrm{End}_F(W)$ with desired derivation-property, hence we seek endomorphisms $e_0, .., e_3$ in $\mathrm{End}_A(W)$ with the property that the adjusted map $\overline{\nabla} : \mathbf{V}_W \to \mathrm{End}_F(W)$ defined by $\overline{\nabla}(\delta_i) = \nabla(\delta_i) + e_i$ is A-linear. We see that we have to solve equations in the ring $\mathrm{End}_A(W)$. In the examples above, it turns out if one looks closely that one can

find elements in $\mathrm{End}_A(W)$ by inspection so as to reduce the problem to solve linear equations in the field F. If one does this, one arrives at the following expressions:

$$\overline{\nabla}_{\delta_0} = \delta_0 + A_0 = \tag{2.0.2}$$

$$\delta_0 + \begin{pmatrix} nk+ml-\frac{1}{2}mn & 0 & 0 & 0 \\ 0 & \frac{3}{2}mn-ml-nk & 0 & \\ 0 & 0 & \frac{1}{2}mn+ml-nk & 0 \\ 0 & 0 & 0 & \frac{1}{2}mn+nk-ml \end{pmatrix}.$$

$$\overline{\nabla}_{\delta_1} = \delta_1 + \begin{pmatrix} 0 & b_2 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & b_3 & 0 \end{pmatrix} = \delta_1 + A_1, \tag{2.0.3}$$

with $b_1 = \frac{1}{4}(mn-2nk-2ml)x^{k-1}y^{l-1}$, $b_2 = \frac{1}{4}(3mn-2ml-2nk)x^{m-k-1}y^{n-l-1}$, $b_3 = \frac{1}{4}(2nk-mn-2ml)x^{m-k-1}y^{l-1}$ and $b_4 = \frac{1}{4}(2nk-2ml+mn)x^{k-1}y^{n-l-1}$.

$$\overline{\nabla}_{\delta_2} = \delta_2 + \begin{pmatrix} 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_4 \\ c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \end{pmatrix} = \delta_2 + A_2, \tag{2.0.4}$$

with $c_1 = \frac{1}{n}(\frac{1}{2}mn-ml-nk)x^{k-1}$, $c_2 = \frac{1}{n}(\frac{3}{2}mn-ml-nk)x^{m-k-1}$, $c_3 = \frac{1}{n}(\frac{1}{2}mn+ml-nk)x^{m-k-1}$ and $c_4 = \frac{1}{n}(ml-nk-\frac{1}{2}mn)x^{k-1}$.

$$\overline{\nabla}_{\delta_3} = \delta_3 + \begin{pmatrix} 0 & 0 & 0 & d_4 \\ 0 & 0 & d_3 & 0 \\ 0 & d_2 & 0 & 0 \\ d_1 & 0 & 0 & 0 \end{pmatrix} = \delta_3 + A_3, \tag{2.0.5}$$

where $d_1 = \frac{1}{m}(\frac{1}{2}mn-ml-nk)y^{l-1}$, $d_2 = \frac{1}{m}(ml+nk-\frac{3}{2}mn)y^{n-l-1}$, $d_3 = \frac{1}{m}(\frac{1}{2}mn+ml-nk)y^{l-1}$ and $d_4 = \frac{1}{m}(\frac{1}{2}mn-ml+nk)y^{n-l-1}$.

Theorem 2.1. *For all $1 \leq k \leq m$ and $1 \leq l \leq n$ the equations 2.0.2-2.0.5 define algebraic $\mathbf{V}_W$-connetions $\overline{\nabla}^{\phi,\psi} : \mathbf{V}_W \to \mathrm{End}_F(W)$ where $W = W(\phi,\psi)$.*

Proof. The module W is given by the exact sequence

$$\cdots \to^{\psi} A^4 \to^{\phi} A^4 \to W \to 0,$$

hence an element w in W is an equivalence class $\overline{a}$ of an element a in A^4. Let a in A^4 be the element

$$a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

and consider the class $w = \overline{a}$ in $W = A^4/im\phi$. Define the $\mathbf{V}_W$-connection $\overline{\nabla}$ as follows:

$$\overline{\nabla}(\delta_i)(w) = (\delta_i + A_i)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}.$$

Then one verifies that this definition gives a well-defined A-linear map $\overline{\nabla}^{\phi,\psi} = \overline{\nabla}$: $\mathbf{V}_W \to \mathrm{End}_F(W)$, and we have proved the assertion. $\square$

Note that the connections $\overline{\nabla}^{\phi,\psi}$ from Theorem 2.1 exist over any field F of characteristic prime to m and n.

In Theorem 2.1 we saw examples where $\mathbf{V}_W$ was the whole module of derivations for a class of modules on some hypersurface singularities. In [19], Theorem 5.1 the splitting type of the principal parts $\mathcal{P}^k(\mathcal{O}(d))$ is calculated on the projective line over a field of characteristic zero. When $d \geq 1$, he following formula is shown:

$$\mathcal{P}^1(\mathcal{O}(d)) \cong \mathcal{O}(d-1) \oplus \mathcal{O}(d-1).$$

It follows that the Atiyah sequence (see [2])

$$0 \to \Omega^1 \otimes \mathcal{O}(d) \to \mathcal{P}^1(\mathcal{O}(d)) \to \mathcal{O}(d) \to 0$$

does not split hence $\mathcal{O}(d)$ does not have a connection

$$\nabla : \mathcal{O}(d) \to \Omega^1 \otimes \mathcal{O}(d).$$

It follows that there does not exist an action

$$\rho : T_{\mathbf{P}^1} \to \mathrm{End}(\mathcal{O}(d)),$$

hence we see that for $\mathcal{O}(d)$ on $\mathbf{P}^1$ over a field of characteristic zero, the linear Lie-Rinehart algebra $\mathbf{V}_{\mathcal{O}(d)}$ is a strict sub-sheaf of the tangent sheaf $T_{\mathbf{P}^1}$.

3. Chern-classes

In this section we define for any F-algebra A where F is any field, and any A-module W of finite presentation with a $\mathbf{V}_W$-connection the first Chern-class $c_1(W)$ in $\mathrm{H}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$, where U in $\mathrm{Spec}(A)$ is the open subset where W is locally free, and $\mathrm{H}^i(\mathbf{V}_W|_U, \mathcal{O}_U)$ is the Chevalley-Hochschild cohomology of the restricted Lie-Rinehart algebra $\mathbf{V}_W|_U$ with values in the sheaf $\mathcal{O}_U$.

Definition 3.1. Let $\mathfrak{g}$ be a Lie-Rinehart algebra and W a $\mathfrak{g}$-module. The Chevalley-Hochshild complex $\mathrm{C}^\bullet(\mathfrak{g}, W)$ is defined as follows:

$$\mathrm{C}^p(\mathfrak{g}, W) = \mathrm{Hom}_A(\mathfrak{g}^{\wedge p}, W),$$

with differentials $d^p : \mathrm{C}^p(\mathfrak{g}, W) \to \mathrm{C}^{p+1}(\mathfrak{g}, W)$ defined by

$$d^p\phi(g_1 \wedge \cdots \wedge g_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} g_i \phi(g_1 \wedge \cdots \wedge \overline{g_i} \wedge \cdots \wedge g_{p+1}) +$$
$$\sum_{i<j} (-1)^{i+j} \phi([g_i, g_j] \wedge \cdots \wedge \overline{g_i} \wedge \cdots \wedge \overline{g_j} \wedge \cdots \wedge g_{p+1}).$$

Here $g\phi(g_1 \wedge \cdots \wedge g_p) = \nabla(g)\phi(g_1 \wedge \cdots \wedge g_p)$ and overlined elements should be excluded. The cohomology $\mathrm{H}^i(\mathrm{C}^\bullet(\mathfrak{g}, W))$ is denoted $\mathrm{H}^i(\mathfrak{g}, W)$.

Note that $\mathrm{C}^\bullet(\mathfrak{g}, W)$ is a complex if and only if the $\mathfrak{g}$-connection ∇ is *flat*, i.e a morphism of Lie-algebras. The complex $\mathrm{C}^\bullet(\mathfrak{g}, W)$ is a complex generalizing simultaneously the algebraic deRham-complex and Chevalley-Eilenberg complex. Now given a $\mathbf{V}_W$-connection ∇ on W, we immediately get a $\mathbf{V}_W$-connection on $\mathrm{End}_A(W)$, denoted $ad\nabla$. We see that if ∇ is flat, it follows that $ad\nabla$ is flat. Consider the open subset U of $\mathrm{Spec}(A)$ where W is locally free. We restrict the $\mathbf{V}_W$-connection to U, to get a connection $\nabla : \mathbf{V}_W|_U \to \mathrm{End}_{\mathcal{O}_U}(W|_U)$. Since $W|_U$ is

locally free we can find an open cover U_i of U where $U_i = D(f_i)$ and $W|_{D(f_i)} = A^n_{f_i}$. On the open subsets U_i we get restricted connections ∇_i, but since $W|_{U_i}$ is a free module, we have on U_i when we pick a basis a flat connection ρ_i. Therefore on any open subset U_i we can consider the complex of sheaves $\mathrm{C}^\bullet(\mathbf{V}_W|_{U_i}, \mathrm{End}_{\mathcal{O}_{U_i}}(W|_{U_i}))$ with respect to the regular connection $ad\rho_i$. We see that $R_\nabla|_{U_i}$ is an element of $\mathrm{C}^2(\mathbf{V}_W|_{U_i}, \mathcal{O}_{U_i})$. One checks that the element $c^i = trace \circ R_\nabla|_{U_i}$ is a cocycle of the complex $\mathrm{C}^\bullet(\mathbf{V}_W|_{U_i}, \mathcal{O}_{U_i})$ for all i. We have constructed elements

$$c^i \in \mathrm{C}^2(\mathbf{V}_W|_{U_i}, \mathcal{O}_{U_i})$$

which coincide on intersections since trace is independent with respect to choice of basis, hence the sheaf-structure on $\mathrm{C}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$ gives a uniquely defined element c in $\mathrm{C}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$, such that $c|_{U_i} = c^i$ for all i. There exists a commutative diagram

$$\begin{tikzcd} \mathrm{C}^p(\mathbf{V}_W|_U, \mathcal{O}_U) \arrow[r, "d^p"] \arrow[d, "|_{U_i}"] & \mathrm{C}^{p+1}(\mathbf{V}_W|_U, \mathcal{O}_U) \arrow[d, "|_{U_i}"] \\ \mathrm{C}^p(\mathbf{V}_W|_{U_i}, \mathcal{O}_{U_i}) \arrow[r, "d^p"] & \mathrm{C}^{p+1}(\mathbf{V}_W|_{U_i}, \mathcal{O}_{U_i}) \end{tikzcd},$$

which proves that the element c is a cocycle in the complex $\mathrm{C}^\bullet(\mathbf{V}_W|_U, \mathcal{O}_U)$.

Theorem 3.2. *There exists a class $c_1(W)$ in $\mathrm{H}^2(\mathbf{V}_W|_U, \mathcal{O}_U)$ which is independent with respect to choice of $\mathbf{V}_W$-connection.*

Proof. Existence of the class $c_1(W)$ follows from the argument above. Independence with respect to choice of connection is straightforward. □

Note that if the $\mathbf{V}_W$-connection is flat, the first Chern-class $c_1(W)$ is zero. Note also that the construction in this section cam be done with any A-module W of finite presentation with a $\mathfrak{g}$-connection, where $\mathfrak{g}$ is any Lie-Rinehart algebra.

Theorem 3.3. *The $\mathbf{V}_W$-connections $\overline{\nabla}^{\phi,\psi}$ calculated in Theorem 2.1 are flat hence $c_1(W(\phi,\psi)) = 0$ for all the modules $W(\phi,\psi)$ on the singularities B_{mn2}.*

Proof. Easy calculation. □

Note that the flat $\mathbf{V}_W$-connections in Theorem 3.3 give rise to a class of left modules on the algebra of differential operators $\mathrm{D}(A)$ where A is the ring $k[x,y,z]/f$ and $f = x^m + y^n + z^2$. Note also that Kohno has in [13] computed the Alexander-polynomial of an irreducible plane curve C in $\mathbf{C}^2$ using a certain logarithmic deRham-complex $\Omega^\bullet_{\mathbf{C}^2}(*C)$. It would be interesting to see if the Alexander-polynomial can be computed in terms of a $\mathfrak{g}$-connection.

4. Surface quotient-singularities

In this section we consider maximal Cohen-Macaulay modules on quotient singularities of dimension two, and their first Chern-class. Let now $F = \mathbf{C}$ be the complex numbers, and let $G \subseteq GL(2,F)$ be a finite sub-group with no pseudo-reflections. Consider the natural action $G \times F^2 \to F^2$, and the quotient $X = F^2/G$. It is an affine scheme with an isolated singularity. Pick a representation $\rho : G \to GL(V)$ where V is an F-vectorspace, and consider the A-module $V \otimes_F A$ where $A = F[x,y]$. Define a G-action as follows: $g(v \otimes a) = \rho(g)v \otimes ga$, and let $W_\rho = (V \otimes_F V)^G$ be the G-invariants of the G-action defined. Then by [9] W_ρ is a maximal Cohen-Macaulay

module on A^G. If ρ is indecomposable, M_ρ is irreducible. The McKay correspondence in dimension 2 says that all maximal Cohen-Macaulay modules on A^G arise this way. Define a G-action on $\mathfrak{g} = \mathrm{Der}_F(A)$ as follows: $(g\delta)(a) = g\delta(g^{-1}a)$, and let $\mathfrak{g}^G$ be the G-invariant derivations. The module $V \otimes_F A$ is a free A-module, hence there exists trivially a regular $\mathfrak{g}$-connection on $V \otimes_F A$. This implies that we get an induced $\mathfrak{g}^G$-connection on $V \otimes_F A$.

Proposition 4.1. *There exists an action of $\mathfrak{g}^G$ on W_ρ.*

Proof. Straightforward. □

From [24] it follows that $\mathfrak{g}^G$ is isomorphic to $\mathrm{Der}_F(A^G)$, hence we have proved that all maximal Cohen-Macaulay modules W_ρ on $\mathrm{Spec}(A^G)$ posess a $\mathrm{Der}_F(A^G)$ -connection, and these are all regular connections since they are induced by the trivial one on $V \otimes_F A$.

Theorem 4.2. *Let $X = \mathrm{Spec}(A^G)$ be a 2-dimensional quotient singularity and let W_ρ be a maximal Cohen-Macaulay module on X then $c_1(W_\rho) = 0$.*

Proof. Follows from the argument above. □

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